Innocent Exclusion is not Contradiction Free

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1. Introduction

Fox (2007) introduces an elegant notion, which he dubs Innocent Exclusion, to state the semantics of exhaustive operators. In later work, Fox (2009) claims that an exhaustive operator stated in terms of Innocent Exclusion will never fall into inconsistency; he calls it 'contradiction free.' In this squib I observe that there are cases in which Innocent Exclusion leads to contradiction. The cases are those studied by Fox & Hackl (2006).

Fox & Hackl (2006) (F&H) observe that a sentence like (1)a does not give rise to the scalar expected implicature that it is not the case that five students smoke. They account for this by arguing that the measurement domain, which contains the denotation of **three**, is densely ordered.

(1) a. More than THREE students smoke
b. $\lambda w.|\mathbf{students'}_w \cap \mathbf{smoke'}_w|>3$

Following Fox (2003), F&H assume that implicatures are introduced by a covert exhaustivity (EXH) operator. The "focus" of the alternative sensitive EXH is the numeral **three**.

(2) EXH[**more than <u>three</u> students smoke**]

Given the semantics of EXH in (3), (2) is undefined.

(3) a. $[\![\text{EXH}]\!](\phi)(d)(w) \Leftrightarrow d = \text{MAX}_{inf}(\phi)(w)$
b. $\text{MAX}_{inf}(\phi)(w) = (\iota x)[\phi(x)(w)=1 \text{ and } \forall y[\phi(y)(w)=1 \rightarrow \phi(x) \text{ entails } \phi(y)]$

The function MAX_{inf} picks out the most informative of a set of degrees. The reason that (2) is undefined is that there is no most informative degree in this case. If the domain of measurement is dense and the number of students that smoke is greater than three, then three is not the most informative degree. Given the density of the domain there is some degree **d** between three and the number of students that smoke. It is true that more than **d** students smoke, but this is not entailed by the proposition that the number of smoking students greater than three.

In independent work, Fox (2007) proposes another way to interpret EXH. In that paper Fox is concerned with implicatures associated with disjunction. For reasons that need not concern us here, he finds it necessary to include each of the disjuncts among the relevant alternatives to the disjunction.

This leads to a problem in the calculation of the implicatures of unembedded disjunctions. Since both p and q entail $p \vee q$, if both are considered as alternatives to $p \vee q$, then by standard scalar reasoning we derive the implicatures $\neg p$ and $\neg q$. But these implicatures contradict the assertion. Building on previous work, notably that of Sauerland (2005), Fox suggests that the problem can be avoided with the notion of *Innocent Exclusion*. Fox proposes that an alternative to an assertion can be innocently excluded (that is, denied) only if it is included in every way of negating as many alternatives to the assertion as possible without contradicting the assertion. In the case of disjunction, there are two ways of negating as many alternatives as possible that do not contradict the assertion: $\{\neg p, \neg(p \wedge q)\}$ and $\{\neg q, \neg(p \wedge q)\}$. Only $p \wedge q$ is denied in both of these sets, so only $p \wedge q$ is innocently excludable.

Fox's formulation is given in (4) below. A is the set of propositions that are alternatives to p. I-E(p,A) is the set of innocently excludable propositions in A relative to the assertion of p. (5) is the set of alternatives to (1)b, given that there is focus on the numeral.

(4) a. $\text{EXH}(p,A,w) \Leftrightarrow p(w)$ and $\forall q \in \text{I-E}(p,A): \neg q(w)$
b. $\text{I-E}(p,A) := \bigcap\{A' \subseteq A$: A' is a maximal set in A s.t. $A'^{\neg} \cup \{p\}$ is consistent$\}$
c. $A^{\neg} := \{\neg p: p \in A\}$

(5) The alternatives to (1)b =
$\{\lambda w.\ |\mathbf{students'}_w \cap \mathbf{smoke'}_w| > m : m \in [0,\infty)\}$[1]

Given the emphasis on consistency in the definition of the semantics of EXH in (4), it is natural to ask whether or not it is consistent with F&H's contradiction-based account of (1)'s lack of an implicature. Fox (2007, 2009) refers to the EXH defined in (4) as "contradiction free" and suggests that F&H's account needs to be modified to accommodate a contradiction-free EXH.

All I wish to observe in this squib is that "contradiction-free" EXH is not contradiction free and that innocent exclusion is fully consistent with F&H's account of (1). The reason is simple to understand. Innocent exclusion works by intersecting certain selected sets of alternatives. To avoid ending up with a null intersection, the sets of alternatives are forced to be as large as possible. Hence, the definition in (4)b makes reference to the <u>maximal</u> sets whose negations are consistent with the assertion.

(6) A is a maximal set in B s.t. A has p iff A has p and there is no C s.t. $C \subseteq B$ and $A \subset C$ and C has p.

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[1] This is an abuse of the notation $|\cdot|$. F&H's account forces us to think of this as a real-valued measure function. By $[0,\infty)$, I mean the non-negative real numbers.

So, the proper functioning of innocent exclusion depends on the existence of such maximal sets. However, when the set of alternatives is densely ordered there will not be any such maximal sets. Every set of alternatives whose negation is consistent with (1) can be expanded to a larger set whose negation is consistent with (1).

2. Proof

I offer here an informal proof that (2) is a contradiction when EXH is interpreted as in (4).

(1) a. More than THREE students smoke
b. $\lambda w.|\mathbf{students}'_w \cap \mathbf{smoke}'_w| > 3$

(2) EXH[**more than <u>three</u> students smoke**]

<u>Abbreviations</u>
A1. $\mathbf{a_n} := \lambda w.|\mathbf{students}'_w \cap \mathbf{smoke}'_w| > n$
A2. $\mathbf{A} := \{\mathbf{a_n} : n \in [0,\infty)\}$
A3. $\mathbf{maxcon_3}(\mathbf{A}) := \{X: X \subseteq A$ and $\{\mathbf{a_3}\} \cup X^{\neg}$ is consistent and there is no Y s.t. $Y \subseteq A$ and $\{\mathbf{a_3}\} \cup Y^{\neg}$ is consistent and $X \subset Y\}$

Part I. I begin by showing that $\mathbf{maxcon_3}(\mathbf{A})$ is empty. The proof is by reductio. We begin by assuming that $\mathbf{maxcon_3}(\mathbf{A})$ is non-empty. I show that this leads to a contradiction.[2]

<u>Definitions</u>
D1. x is a <u>lower bound</u> of a set S iff for all $y \in S$, $x \leq y$
D2. x is the <u>greatest lower bound</u> of a set S of real numbers (glb(S)) iff x is a lower bound of S and for all $y \neq x$ s.t. y is a lower bound of x, $x > y$.
D3. For any subset B of **A**,
$glb_B := glb(\{m: \mathbf{a_m} \in B\})$

<u>Proposition</u>
P1. Every (non-empty) set of real numbers (that has a lower bound) has a glb. (Dedekind)

P2. Every (non-empty) subset X of **A** has a glb_X. (follows from P1 and D3)

<u>Hypothesis</u>
H1. Suppose for reductio that $C \in \mathbf{maxcon_3}(\mathbf{A})$

[2] The proof makes use of the continuity of the real numbers. This is not necessary. The proof goes through for the rationals, as well. Rather than referring to the glb, the proof proceeds by quantifying over lower bounds. The proof with the reals is simpler.

I will now show that, if H1 is true, C has no glb_C – contradicting P2; P2 entails that C has a greatest lower bound ($\mathbf{glb_C}$).

Theorem

T1. $3 < \mathbf{glb_C}$ or $3 = \mathbf{glb_C}$ or $3 > \mathbf{glb_C}$ [follows from property of real number line]

Case (i): $\mathbf{glb_C} < 3$

Case (ia): $\mathbf{a_{glb_C}} \in C$

$\neg \mathbf{a_{glb_C}}$ is inconsistent with $\mathbf{a_3}$. ↯[3]

Case (ib) $\mathbf{a_{glb_C}} \notin C$

By density, $\exists d[\ \mathbf{glb_C} < d < 3\ ]$

Reductio: suppose there is no x between $\mathbf{glb_C}$ and 3 s.t. $x \in C$

But then $\mathbf{glb_C}$ is not the greatest lower bound of C. ↯

Hence , there is a number $\mathbf{e}$ between $\mathbf{glb_C}$ and 3 s.t. $\mathbf{e} \in C$

$\neg \mathbf{a_e}$ is inconsistent with $\mathbf{a_3}$. ↯

Case (ii): $\mathbf{glb_C} = 3$

Case (iia): $\mathbf{a_{glb_C}} \in C$

$\neg \mathbf{a_{glb_C}}$ is inconsistent with $\mathbf{a_3}$. ↯

Case (iib) $\mathbf{a_{glb_C}} \notin C$ [This is F&H's result]

$\mathbf{a_3}$ entails that $|\mathbf{students'} \cap \mathbf{smoke'}| = \mathbf{n}$ for some real $\mathbf{n}$

By density, $\exists d[\ 3 < d < n\ ]$

Reductio: Suppose there is no x between 3 and n s.t. $x \in C$

But then 3 is not the greatest lower bound of C. ↯

Hence, there is an $\mathbf{e}$ between 3 and $\mathbf{n}$ s.t. $\mathbf{e} \in C$

But $\neg \mathbf{a_e}$ contradicts $|\mathbf{students'} \cap \mathbf{smoke'}| = \mathbf{n}$. ↯

Case (iii): $\mathbf{glb_C} > 3$

Case (iiia): $\mathbf{a_{glb_C}} \in C$

By density, $\exists d[\ 3 < d < \mathbf{glb_C}\ ]$

$C \cup \{\neg \mathbf{a_d}\}$ is consistent with $\mathbf{a_3}$. ↯ (C is not maximal)

Proof: $\neg \mathbf{a_d}$ entails every member of C

$\{\neg \mathbf{a_d}, \mathbf{a_3}\}$ is consistent.

Case (iiib) $\mathbf{a_{glb_C}} \notin C$

$C \cup \{\neg\ \mathbf{a_{glb_C}}\}$ is consistent with $\mathbf{a_3}$. ↯ (C is not maximal)

So, there is no X such that $X \in \mathbf{maxcon_3(A)}$, i.e., H1 is false.

Thus, $\mathbf{maxcon_3(A)} = \emptyset$

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[3] I use the symbol ↯ to indicate that a contradiction has been derived.

Part II. Having shown that $\mathbf{maxcon_3}(\mathbf{A})$ is empty, I now show that (2) is a contradiction. First, given the definition of $\bigcap$:[4]

$$\bigcap\mathbf{maxcon_3}(\mathbf{A}) = \bigcap\varnothing = \text{the entire domain of propositions } (D_{<s,t>}).$$

The truth conditions for (2) under Fox's (2007) theory are below:

$$\text{EXH}(\mathbf{a_3},\mathbf{A},w) \Leftrightarrow \mathbf{a_3}(w) \text{ and } \forall q \in \text{I-E}(\mathbf{a_3},\mathbf{A}){:}\neg q(w)$$

These are contradictory given that

$\text{I-E}(\mathbf{a_3},\mathbf{A}) = D_{<s,t>}$ and [recall $\text{I-E}(\mathbf{a_3},\mathbf{A}) = \bigcap\mathbf{maxcon_3}(\mathbf{A})$]
$D_{<s,t>}$ contains $\mathbf{a_3}$ and pairs of propositions that contradict each other.

Hence (2) is a contradiction.

QED

3. Discussion

This result shows that even if the covert exhaustivity operator EXH makes use of innocent exclusion in its semantics, as suggested by Fox (2007), contradictions can still be derived. For example, contradictions still arise in the cases analyzed by Fox & Hackl (2006). In particular, whenever Innocent Exclusion EXH applies in cases like (2) and a dense scale is assumed, the result is a contradiction. Consequently, the representation with the optional EXH operator is discarded and an EXH-less LF is selected. This latter LF, of course, carries no implicature – in accord with Fox & Hackl's observation concerning (1). This suggests that both Fox (2007) and Fox & Hackl (2006) can be maintained in their essential details, contrary to the claim made in Fox (2009).

References

[4] The reader may recall that in Zermelo-Frankel set theory the intersection of the empty set is undefined. The reason is that in ZF there is no universal class. We are on safe ground, however, if we restrict the domain against which n-ary intersection is defined.

Thus, I take $\bigcap$ to be an instance of a polymorphic operation of n-ary intersection, defined for all 'set' types:

$\bigcap_{<x,t>} := \lambda F_{<<x,t>,t>}.\lambda p_x.$ for all $G \in D_{<x,t>}$ s.t. $F(G)=1$, $G(p)=1$

If F maps all members of $D_{<x,t>}$ to 0, then $\bigcap F$ is the function with domain D_x that maps all members of D_x to 1.

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